Information Topologies on Non-Commutative State Spaces

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Information Geometry for Non-Maximal Rank

An information topology extends the Pythagorean and Projection Theorem of quantum information geometry to states of non-maximal rank. The norm topology does it wrongly.

- These theorems provide a geometric framework for the MaxEnt map, short for maximum-entropy inference map under linear constraints (Boltzmann 1877, Jaynes 1957)
- Constraints on marginals define many-party correlation quantities, known as irreducible correlation (Amari 2001, Linden et al. 2002, Zhou 2008, Ay et al. 2011)
- Failure of norm extendability means that the MaxEnt map can be discontinuous for finite-dimensional Hilbert spaces, in non-commutative C*-algebras (W and Knauf 2012)

From Information Geometry to Phase Transitions

Extended theorems apply to states of non-maximal rank, for example to ground states.

- Variation and discontinuity of the MaxEnt map is a signature of quantum phase transitions (Arrachea et al. 1992, Chen et al. 2015)
- Detection of topological order in ground states with irreducible correlation (Kato et al. 2016, Liu et al. 2016)
- Non-analyticity of the ground state energy of a one-parameter Hamiltonian creates discontinuities of a MaxEnt map (Leake et al. 2014 and W 2016)

Definitions

C*-algebra M_n of n-by-n matrices, ordering $b \succeq a$ means that $b-a$ is positive semi-definite, Hilbert-Schmidt inner product $\langle a,b\rangle := \mathrm{tr}(a^*b)$, $a,b\in M_n$. Hermitian matrices $M_n^{\mathrm{h}} := \{a\in M_n \mid a^*=a\}$, **state space** $\mathcal{D}_n := \{\rho\in M_n \mid \rho\succeq 0, \mathrm{tr}(\rho)=1\}$ of M_n. The **support projection** $s(a)$ of $a\in M_n^{\mathrm{h}}$ is the projection onto the range of a. The **relative entropy**, an asymmetric distance, is defined by $D:\mathcal{D}_n\times\mathcal{D}_n\to[0,\infty]$,

$$D(\rho,\sigma) := \begin{cases}\langle\rho,\log(\rho)-\log(\sigma)\rangle, & \text{if } s(\sigma)\succeq s(\rho)\\ +\infty, & \text{else}\end{cases} \qquad \rho,\sigma\in\mathcal{D}_n.$$

The **entropy distance** of $\rho\in\mathcal{D}_n$ from $X\subset\mathcal{D}_n$ is $D(\rho,X) := \inf_{\sigma\in X} D(\rho,\sigma)$. The set of density matrices of full support is denoted $\mathcal{D}_n^+ := \{\rho\in\mathcal{D}_n \mid s(\rho)=\mathbb{1}\}$.

Elementary Information Geometry

Information geometry studies differential manifolds whose points are probability vectors (Amari 1987) or quantum states (Petz 1994, Nagaoka 1995).

Pythagorean Theorem for Curves

Let $\rho,\sigma,\tau\in\mathcal{D}_n$, $\sigma,\tau\in\mathcal{D}_n^+$, and $\langle\rho-\sigma,\log(\tau)-\log(\sigma)\rangle=0$. Then

$$D(\rho,\sigma)+D(\sigma,\tau)=D(\rho,\tau).$$

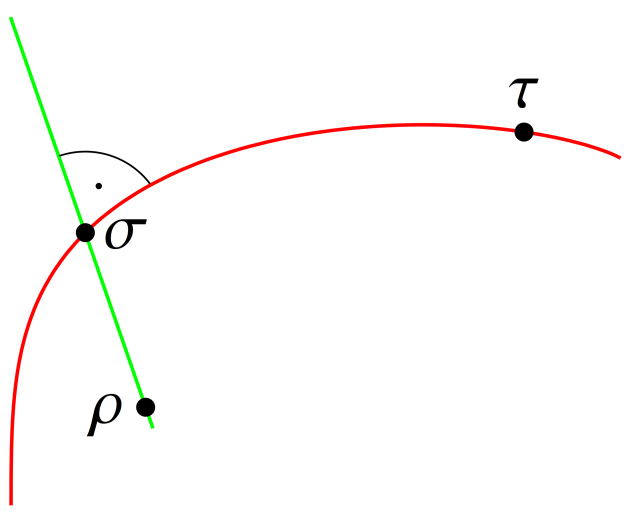


Figure:
(-1)-geodesic $\lambda\mapsto\sigma+\lambda(\rho-\sigma)$
$(+1)$-geodesic $\mu\mapsto R[\log(\sigma)+\mu(\log(\tau)-\log(\sigma))]$
with $R(a)=e^a/\mathrm{tr}(e^a)$

Proof: $D(\rho,\tau)-D(\rho,\sigma)-D(\sigma,\tau)=\langle\rho,\log(\sigma)-\log(\tau)\rangle-\langle\sigma,\log(\sigma)-\log(\tau)\rangle=0.$ □

Define an **exponential family** $\mathcal{E}=\mathcal{E}(U)=\{R(u)\mid u\in U\}$, $U\subset M_n^{\mathrm{h}}$ a linear subspace.

Pythagorean Theorem

Let $\rho,\sigma,\tau\in\mathcal{D}_n$, $\sigma,\tau\in\mathcal{E}(U)$, and $\langle\rho-\sigma,U\rangle=0$. Then

$$D(\rho,\sigma)+D(\sigma,\tau)=D(\rho,\tau).$$

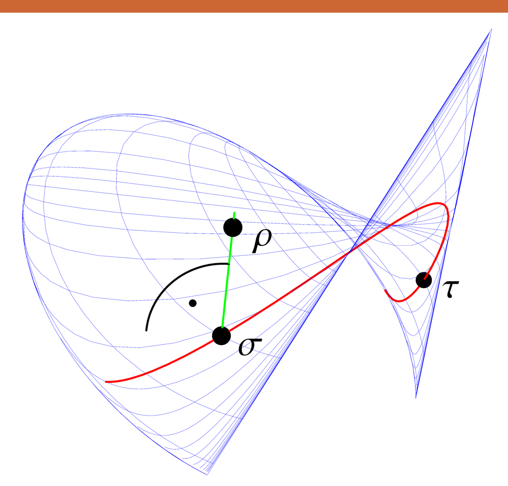


Figure: Two-dimensional exponential family $\mathcal{E}(U)$, (+1)-geodesics (blue, red), and a (-1)-geodesic

Proof: Use $\langle\rho-\sigma,\log(\tau)-\log(\sigma)\rangle=0$ and the Pythagorean Theorem for Curves. □

Projection Theorem

Let $\rho\in(\mathcal{E}(U)+U^\perp)\cap\mathcal{D}_n$.

- There exists a unique state $\pi_{\mathcal{E}}(\rho)\in\mathcal{E}$ such that $\rho-\pi_{\mathcal{E}}(\rho)\perp U$.
- We have $D(\rho,\mathcal{E})=\min_{\sigma\in\mathcal{E}}D(\rho,\sigma)=D(\rho,\pi_{\mathcal{E}}(\rho))$.

Proof: Use the Pythagorean Theorem. □

Why Topology, and Which Topology?

The Projection Theorem shows that $\mathcal{E}(U)$ is parametrized by its projection $\pi_U(\mathcal{E}(U))$ where $\pi_U : M_n^{\mathrm{h}}\to U$ denotes orthogonal projection onto U. Wichmann's Theorem suggests to extend the Pythagorean and Projection Theorems to the norm closure of $\mathcal{E}(U)$.

E. H. Wichmann's Theorem (JMP 4, 884, 1963)

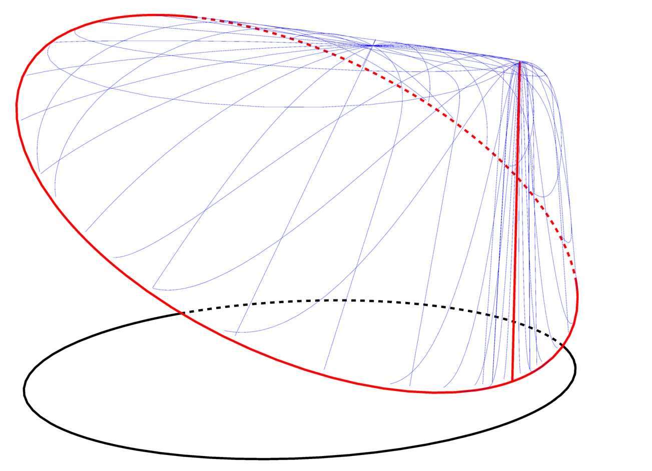
The norm closure $\overline{\pi_U(\mathcal{E}(U))}$ is the projection $\pi_U(\mathcal{D}_n)$ of the state space $\mathcal{D}_n$ of M_n.

The convex set $\pi_U(\mathcal{D}_n)$ is isomorphic to the **state space** of the operator system $U+\mathrm{i}U+\mathbb{C}\mathbb{1}$ (W 2017) and to a **joint algebraic numerical range** (Müller 2010).

Failure of the Norm Closure

Figure: Two-dimensional exponential family $\mathcal{E}(U)$
- $(+1)$-geodesics in $\mathcal{E}(U)$
- components of $\overline{\mathcal{E}(U)}$ outside of $\mathcal{E}(U)$
- **boundary** of $\pi_U(\mathcal{D}_n)$

The vertical segment projects to a single point in $\pi_U(\mathcal{D}_n)$, only its top end is a maximum-entropy state (W and Knauf, JMP **53**, 102206, 2012).



An information closure resolves the problem (W, Journal of Convex Analysis **21**, 339, 2014).

The rI-Closure

The **rI-closure** of $X\subset\mathcal{D}_n$ is $\mathrm{cl}^{\,\mathrm{rI}}(X) := \{\rho\in\mathcal{D}_n \mid D(\rho,X)=0\}$.

- the acronym **rI** stands for **reverse information** (Csiszár and Matúš 2003)
- the rI-closure is the closure of the **rI-topology** which is generated by the open disks $\{\sigma\in\mathcal{D}_n \mid D(\rho,\sigma)<\epsilon\}$, $\rho\in\mathcal{D}_n$, $\epsilon\in(0,+\infty]$; caution: this is wrong for normal states of the commutative algebra $\ell^\infty(\mathbb{Z})$, that is for probability measures on $\mathbb{Z}$ (Csiszár 1964)
- $O\subset\mathcal{D}_n$ is open in the rI-topology if for all $\sigma\in O$ and all sequences $(\sigma_i)_{i\in\mathbb{N}}\subset\mathcal{D}_n$ such that $\lim_{i\to\infty}D(\sigma,\sigma_i)=0$ there is $N\in\mathbb{N}$ such that for all $i\geq N$ we have $\sigma_i\in O$; analogous topologies are known in probability theory (Dudley 1998, Harremoës 2002)
- by Pinsker's inequality $\|\rho-\sigma\|_1^2\leq 2D(\rho,\sigma)$, $\rho,\sigma\in\mathcal{D}_n$, the rI-topology is finer than the norm topology; equality holds only for commutative subalgebras of M_n

Theorems About rI-Closures

Transversality

If $\rho\in\mathcal{D}_n$ then there is a unique state $\pi_{\mathcal{E}}(\rho)\in\mathrm{cl}^{\,\mathrm{rI}}(\mathcal{E}(U))$ such that $\langle\rho-\pi_{\mathcal{E}}(\rho),U\rangle=0$.

Pythagorean Theorem

If $\rho,\sigma\in\mathcal{D}_n$ and $\sigma\in\mathrm{cl}^{\,\mathrm{rI}}(\mathcal{E})$ then $D(\rho,\pi_{\mathcal{E}}(\rho))+D(\pi_{\mathcal{E}}(\rho),\sigma)=D(\rho,\sigma)$.

Projection Theorem

If $\rho\in\mathcal{D}_n$ then $D(\rho,\mathcal{E})=D(\rho,\mathrm{cl}^{\,\mathrm{rI}}(\mathcal{E}))=D(\rho,\pi_{\mathcal{E}}(\rho))$.

Convex geometry and ground spaces are used in the proofs (W, *ibidem*, 2014).

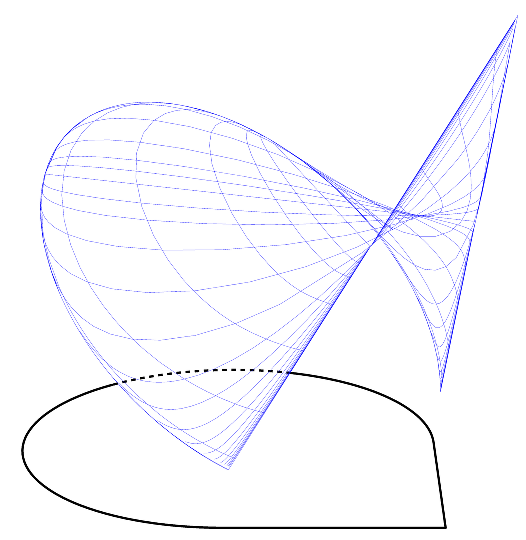
Exposed Faces and Ground Space Projections

Figure: Two-dimensional exponential family $\mathcal{E}(U)$
- $(+1)$-geodesics in $\mathcal{E}(U)$
- **boundary** of $\pi_U(\mathcal{D}_n)$

An **exposed face** of $\pi_U(\mathcal{D}_n)$ is a subset of minimizers of a linear functional,

$$F(u) := \mathrm{argmin}\{\langle z,u\rangle \mid z\in\pi_U(\mathcal{D}_n)\}, \qquad u\in U.$$

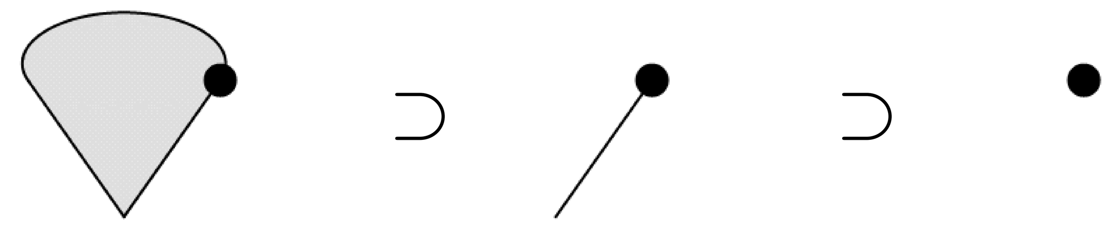
The **ground space projection** $p(u)$ of $u\in U$ is the projection onto the eigenspace of the smallest eigenvalue of u. There is a one-to-one correspondence $F(u)\cong p(u)$.



Poonems are Essential to Prove the Projection Theorem

Ansatz: Cover $\pi_U(\mathcal{D}_n)$ with projections of (+1)-geodesic in $\mathcal{E}(U)$ and their limit points.

Non-exposed points of $\pi_U(\mathcal{D}_n)$ are not reached in one, but in a sequence of limits, labeled by **poonems** (Grünbaum 1966) or **access sequences** (Csiszár and Matúš 2005). By definition, $\pi_U(\mathcal{D}_n)$ is a poonem and every exposed faces of a poonem is a poonem.



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