Continuity of many-party correlations

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Contents

- Maximum-entropy principle and irreducible correlation
- Theorems about continuity (open maps and face function)
- Continuity in 2D and 3D: (joint) numerical range

Quantum systems

- a **quantum system** is described by a C*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ (assume $\dim(\mathcal{H}) < \infty$)
- **observables** / **Hamiltonians** $\mathcal{A}|_{\mathrm{sa}} := \{H \in \mathcal{A} \mid H^* = H\}$
- **quantum states** $\mathcal{M} = \mathcal{M}(\mathcal{A}) = \{\rho \in \mathcal{A} \mid \rho \succeq 0, \mathrm{tr}(\rho) = 1\}$

Many-party systems

- composed of units $[N] := \{1, \ldots, N\}$, $N \in \mathbb{N}$
- **unit algebras** $\mathcal{A}_i$, $i \in [N]$
- **local algebras** $\mathcal{A}_\nu := \bigotimes_{i\in\nu} \mathcal{A}_i$, $\nu \subset [N]$
- embedding into **global algebra** $\mathcal{A}_\nu \hookrightarrow \mathcal{A}_{[N]} = \bigotimes_{i=1}^{N} \mathcal{A}_i$

Maximum-entropy principle (Boltzmann 1877, ...)

- **von Neumann entropy** $S : \mathcal{M} \to \mathbb{R}$, $\rho \mapsto -\operatorname{tr} \rho \log(\rho)$

  quantifies disorder = uncertainty = lack of pureness = lack of information
- subspace $X \subset \mathcal{A}|_{\mathrm{sa}}$, orthogonal projection $\pi : \mathcal{M} \to X$

  with respect to Hilbert-Schmidt inner product $\langle a, b \rangle = \operatorname{tr}(a^* b)$, $a, b \in \mathcal{A}$
- **projection** $\pi(\mathcal{M}) \subset X$

  reduction of statistical model of quantum mechanics (Holevo 1980)
- **inference map** $\rho^* : \pi(\mathcal{M}) \to \mathcal{M}$, $x \mapsto \operatorname{argmax}_{\pi(\rho)=x} S(\rho)$

  chooses a state which is most unbiased, maximal non-committal with regard to missing information (Jaynes 1957)

Missing information in projections

- $I = I(X) : \mathcal{M} \to \mathbb{R}, \rho \mapsto S(\rho^*(\pi(\rho))) - S(\rho)$

  information in $\rho \in \mathcal{M}$ which is missing in $\rho^*(\pi(\rho))$, or with $\rho^*(x) \cong x$

  information in $\rho \in \mathcal{M}$ which is missing in $\pi(\rho)$

Missing information in marginals (LPW'02)

- three qubits $\mathcal{A} = M_2 \otimes M_2 \otimes M_2$ (complex 2-by-2 matrices M_2)
- $X_2 = \text{span}\{\sigma_i \otimes \sigma_j \otimes \sigma_k \mid i, j, k \in \{0, 1, 2, 3\}, i \cdot j \cdot k = 0\}$
  $\sigma_0 = \mathbb{1}, \sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_2 = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right), \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$
- isometry to 2-party marginals
  $\pi(\mathcal{M}) \cong \{(\rho_{AB}, \rho_{BC}, \rho_{AC}) \mid \rho \in \mathcal{M}\}$
- **irreducible 3-party correlation** $I(X_2)(\rho)$ of $\rho \in \mathcal{M}$

  information in ρ which is missing in the two-party subsystem marginals

Remarks on irreducible correlation

- **k-local Hamiltonians** acting on N-party Hilbert space:
  $X_k := \{\sum_{|\nu|=k} H_\nu \mid H_\nu \in \mathcal{A}_\nu\} \subset \mathcal{A}_{[N]}|_{\mathrm{sa}}$
  $I(X_k)(\rho)$ is the information in ρ which is missing in the k-party marginals
- **irreducible k-party correlation** (Z'08)
  $C_k(\rho) = I(X_{k-1})(\rho) - I(X_k)(\rho)$ is the information in the k-party marginals of ρ which is missing in the $(k-1)$-party marginals
- **information geometry** (A'02, AOBJ'11, Z'09, W'14)
  $I(X)$ is the relative entropy distance from an exponential family
- C_k is studied in condensed matter physics (CJLPSYZZ'15, KFM'16, LZZ'16) connection to topological entanglement entropy

Irreducible correlation $I(X_2)$ has discontinuities

Proof 1: Pure state reconstruction

Thm. 1 (LPW'02) For almost every pure state $|x\rangle \in (\mathbb{C}^2)^{\otimes 3}$ exists a unique state in $\mathcal{M}$ whose 2-party marginals are equal to those of $|x\rangle$. The exceptions are local unitarily equivalent to $\alpha|000\rangle + \beta|111\rangle$.

$\Longrightarrow$ $I(X_2) = 0$ almost everywhere while the GHZ state $|\mathrm{GHZ}\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ has $I(X_2)(|\mathrm{GHZ}\rangle) = \log(2)$

$\Longrightarrow$ discontinuity of $I(X_2)$ at $|\mathrm{GHZ}\rangle$

Proof 2: Continuity of the inference map ρ^*

Thm. 2 (W'14) The map $\rho^* : \pi(\mathcal{M}) \to \mathcal{M}$ is continuous at $x \in \pi(\mathcal{M})$ iff $I(X) : \mathcal{M} \to \mathbb{R}$ is continuous on $\pi^{-1}(x)$.

an example of discontinuous ρ^* is $X = \mathrm{span}\{A_1, A_2\}$ for

$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (WK'12)

Thm. 3 (W'14) The map $\rho^* : \pi(\mathcal{M}) \to \mathcal{M}$ is continuous at $x \in \pi(\mathcal{M})$ iff π is open at $\rho^*(x)$.

we say that π is **open** at ρ if π maps neighborhoods of ρ to neighborhoods of $\pi(\rho)$

Thm. 3 $\implies$ sufficient conditions of continuity of ρ^* (e.g. $\pi(\mathcal{M})$ a polytope) the **face function** (KM 1971) of $\pi(\mathcal{M})$ is the set-valued function $F : \pi(\mathcal{M}) \to \pi(\mathcal{M})$ which maps x to the union of all closed segments in $\pi(\mathcal{M})$ containing x in their relative interior.

Thm. 4 (RSSW'16) If a sequence $(x_i)_{i\in\mathbb{N}} \subset \pi(\mathcal{M})$ converges to $x \in \pi(\mathcal{M})$ and $\rho^*(x_i)_{i\in\mathbb{N}}$ converges to $\rho^*(x)$ then $\dim F(x) \leq \liminf_{i\in\mathbb{N}} \dim F(x_i)$ holds.

$\implies$ sufficient condition for discontinuity of ρ^*

Proof 2 (summary). $\pi(|\mathrm{GHZ}\rangle)$ is no extreme point of $\pi(\mathcal{M})$ because $|\mathrm{GHZ}\rangle_{\mathrm{AB}} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$ is mid-point between $|000\rangle_{\mathrm{AB}} = |00\rangle$ and $|111\rangle_{\mathrm{AB}} = |11\rangle$. There exists a curve of extreme points of $\pi(\mathcal{M})$ converging to $\pi(|\mathrm{GHZ}\rangle)$.
Thms 2,3,4 $\implies$ discontinuity of $I(X_2)$ on $\pi^{-1}(\pi(|\mathrm{GHZ}\rangle))$

Where are the discontinuities of $I(X_2)$ for three qubits?

A question about subalgebras

$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ from the example (WK'12) generate the (real) C*-subalgebra

$\mathcal{B} = \mathrm{span}\{\mathbb{1}, \sigma_0 \oplus 0, \sigma_1 \oplus 0, \sigma_2 \oplus 0, \mathrm{i}\,\sigma_3 \oplus 0\}$ of $\mathcal{A} = M_3$

the state space $\mathcal{M}(\mathcal{B})$ being a 3D cone simplifies the continuity analysis of ρ^* (compare to $\dim \mathcal{M}(\mathcal{B}) = 8$)

Question Let $\mathcal{B} \subset \mathcal{A}$ be a C*-subalgebra. Is the orthogonal projection $\mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ open?

If $\dim(X) \leq 2$ then $\pi(\mathcal{M}) \cong$ numerical range

Let $X = \mathrm{span}(A_1, A_2)$ for self-adjoint $A_1, A_2 \in M_d$ and put $A = A_1 + \mathrm{i}\, A_2$. Let $S\mathbb{C}^d := \{x \in \mathbb{C}^d \mid \langle x, x \rangle = 1\}$ denote the unit sphere and $f_A : S\mathbb{C}^d \to \mathbb{C}$, $x \mapsto \langle x | A(x) \rangle$. The **numerical range**

$$W(A) = \{f_A(x) \mid x \in S\mathbb{C}^d\}$$

is isometric to $\pi(\mathcal{M})$ (BO'67).

Thm. 5 (RSSW'16) If $d = 3$, then ρ^* is continuous unless A_1, A_2 are the matrices of the example (WK'12) up to unitary similarity and up to replacement of A_1, A_2 with matrices having the same $\mathrm{span}(\mathbb{1}, A_1, A_2)$.

A matrix is **unitary irreducible** if it is not unitary similar to a block diagonal matrix with two proper blocks.

Thm. 6 (RSSW'16) If $d = 4, 5$ and $A_1 + \mathrm{i}\, A_2$ is unitary irreducible then ρ^* has at most $d - 3$ points of discontinuity.

The proof studies extreme points x of $\pi(\mathcal{M})$ where $\pi^{-1}(x)$ is no singleton (**multiply generated**) and which are not the intersection of two segments (**round boundary points**). Those points are isolated in $\pi(\mathcal{M})$ for $d \leq 5$ and A unitarily irreducible. For $d \geq 6$ the whole boundary of $\pi(\mathcal{M})$ may consist of multiply generated round extreme points (LLS'14).

Thm. 7 (SW'16) An extreme point $x \in W(A)$ is multiply generated iff a sequence of flat boundary portions of $A_i \to A$ converges to x in Hausdorff distance.

Equivalent continuity problem

The multi-valued map $f_A^{-1} : W(A) \to S\mathbb{C}^d$ is **strongly continuous** at $\alpha \in W(A)$ if f_A is open at every $x \in f_A^{-1}(\alpha)$ (CJKLS'13, LLS'14).

Thm. 8 (W'16) The map f_A^{-1} is strongly continuous at $\alpha \in W(A)$ iff the inference map ρ^* is continuous at α.

notice:

$f_A^{-1}(\alpha)$ contains minimum-entropy states

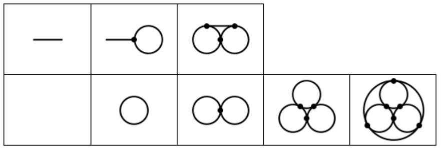
$\leftrightarrow$

$\rho^*(\alpha)$ is a maximum-entropy state

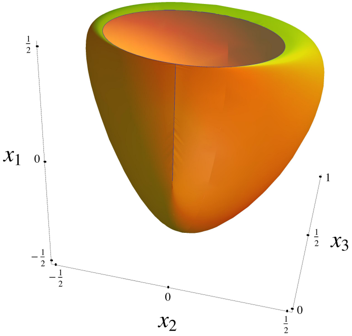
If $\dim(X) \leq 3$ then $\pi(\mathcal{M}) \cong$ joint numerical range ($d \geq 3$)

Let $X = \operatorname{span}(A_1, A_2, A_3)$ for self-adjoint $A_1, A_2 \in M_d$. Classification of $\pi(\mathcal{M})$ for $d = 3$ in terms of the number of segments s and ellipses e which are faces of $\pi(\mathcal{M})$.

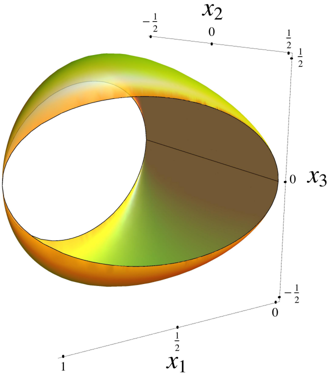
Thm. 8 (SWZ'16) Let $d = 3$, $\dim(\pi(\mathcal{M})) = 3$, and let $\pi(\mathcal{M})$ have no 3D normal cone. Then the following configurations of segments and ellipses are possible:



Two 3D examples with discontinuous ρ^*



(CN'10)



(CJLPSYZZ'15)

Thanks for your attention

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